Stochastic Sensitivity Methods and Application to the Shipboard Power System

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Abstract
In large-scale dynamic systems, including power systems, sensitivity analysis can provide a powerful tool in ranking the importance of random input channels and quantifying their interactions. In this work, we propose new methods for stochastic sensitivity analysis and compare them against the Morris method for stationary and time-dependent problems. We then apply the most efficient of these methods (a combination of Quasi-Monte Carlo and Morris methods) to a subsystem of an all-electric ship model with multi-rate dynamics, consisting of an AC power distribution with an open-loop propulsion system. In a startup transient, we identify the most important parameters in the short-time regime, where the electrical components dominate, and in the long-term regime where the mechanical components dominate.

Index Terms—Sensitivity, Monte Carlo methods, Marine vehicle power systems.

I. INTRODUCTION
One of the primary goals in the design of the future electric ship is to minimize the number of crew and to increase system monitoring and control automation, for reduction of operational costs and increased reliability [23]. However, to maintain situational awareness and fault tolerance, a complete understanding of the sensitivities of the system has to be obtained. In particular, the impact of stochastic parameter variations, and the potential interactions of these variations are of interest.

Sensitivity analysis has been studied from two different perspectives – ranking the inputs’ significance and evaluating the outputs’ tolerance limit [3] – when the inputs are known to vary within a specified range. Only the first approach, the main focus in this study, provides information on which parameters and inputs have strongest influence on the system outputs and on how strong is the coupling or interaction among parameters. Two main classes of techniques for ranking these inputs in sensitivity studies are local and global methods. The local approach [8], [9], [17], which relies on a partial derivative of outputs with respect to input parameters, is used to measure the sensitivity around a local operating point. When the system has strong nonlinearities and the parameters fluctuate within a wide range from their nominal values, local sensitivity does not provide accurate information to the designer. On the other hand, the global approach examines the sensitivity from the entire range of the parameter variations. Global screening methods, based on One-At-a-Time perturbation of parameters, successfully rank the significant parameters and identify interactions. Several screening methods have been proposed in the literature, for example, the Morris method [5], [14], [19], Cotter’s method [4], factorial experimentation [2], iterated fractional factorial design [18], and variance-base sensitivity measure [5], [19]; different techniques have their strengths and weaknesses. The Morris method scales well to large systems. Only the worst-case analysis of a system is examined for the upper and lower bounds of inputs in Cotter’s method. In factorial experimentation, all combinations of the parameters’ interactions are evaluated at the same time, requiring intensive computation. Iterated fractional factor design reduces this large input-combination computation by evaluating only the most important combinations. As a result, however, the sensitivity indices might be biased. Because of a combinatorial evaluation of hundreds or thousands parameters, the advantages of the variance sensitivity measure is overcome by its time-consuming characteristics.

Other global sensitivity analysis techniques that provide ranking include regression analysis [19] and ANOVA decomposition [20]. Both of these are limited because of the required prior knowledge about the structure of the system, which is not applicable to general functions or systems. Trajectory sensitivity, based on perturbations of transient trajectories, is another technique for sensitivity analysis [1], [7].

In this paper, an overview of sensitivity analysis methods is described in Section 2 with the technical details given in Appendix I. In Section 3, a comparison of all methods for a test nonlinear function and a test system of Ordinary Differential Equations (ODEs) are presented. Using these sensitivity techniques, the parametric sensitivity and interactions are then obtained for an AC power generation and open-loop propulsion system in Section 4.

II. SENSITIVITY ANALYSIS METHODS
Practically speaking, when a large-scale system has hundreds or thousands of input parameters \([x_1, \ldots, x_d]\), it can be impossible to fully investigate the effects of all combinations. The goal of parameter screening is to determine which inputs have the most effect on the outputs and to rank the inputs
accordingly, so that further experiments or calculations can focus only on the most sensitive channels.

A key quantity is the elementary effect of the $i$'th input on the $j$'th output. $(EE^j_i(t))$ is defined as an approximate gradient, wherein the $i$'th input is varied by the amount $\Delta$, which is not necessarily small. This gradient, which can be a scalar value for a function or a time-varying quantity for ODEs, is calculated at a number of points over the input random domain; and the statistics of this collection are formed. To avoid averaging to zero, we use the absolute difference of the $j$th output $[5]$ in the definition, i.e.,

$$EE^j_i(t) = \frac{|y_j(x_1, \ldots, x_i + \Delta, \ldots, x_d, t) - y_j(t)|}{\Delta}$$

where $x_i$ with $i = 1, \ldots, d$ is contained within the $d$-dimensional domain; the various methods we consider employ different points. Using this local gradient computation, we have intuitively that when $\partial y_j/\partial x_i = 0$, (1) a non-zero constant, or (3) a non-constant function of input parameters, the effects of $x_i$ on $y_j$ are (1) negligible, (2) linear and additive, or (3) nonlinear and/or coupled, respectively. More specifically, the expected value of the elementary effect, $E[EE^j_i]$, with respect to ensembles captures the overall first-order effect of the $i$th input on the $j$th output. The standard deviation $\sigma[EE^j_i]$ represents a combined interaction and nonlinear effect of the $i$th parameter on the $j$th output in the gradient-based method, but in the variance-based method, $\sigma[EE^j_i]$ excludes the nonlinear effect. When we drop the $t$, $EE^j_i$ represents the whole trajectory. The $j$ superscript is dropped when the output channel is indicated specifically. We investigate with computations several gradient-based methods - Morris, Monte Carlo Sampling, and Collocation, as well as a new variance-based method for characterizing sensitivity. Because the main objective of this paper is to apply sensitivity analysis to electric ship models, details of these methods are presented in Appendix I, and are available in more depth in [15].

We note here that Galerkin polynomial chaos [6] can also be employed in computing sensitivities; in this case, we ideally obtain the results analytically from the coefficients of probabilistic modes. However, this method is generally limited to small random dimension (a small number of stochastic input parameters), and certain hard nonlinearities impose undue computational burden. Therefore, most of our simulations were obtained using the probabilistic collocation approach [9], [21], [22]. We could still follow the analytical approach to obtain the sensitivity index but this requires a multi-dimensional projection, which may be prohibitively expensive in high dimensions. The methods, discussed in Appendix A, are general and can be applied to any stochastic simulation results.

In order to compare values of $EE^j_i$ for different output quantities, we normalize $EE^j_i$ with the nominal value of $x_i$ divided by the maximum magnitude of all $y_j$ ensemble. Although a nominal sensitivity involves a scalar $x_i$ and scalar $y_j$, to study the sensitivity over a time interval, we define the average sensitivity as $ES_{2,(j,i)}$ using the $L_2$ norm on time of the ensemble average

$$ES_{2,(j,i)} = \|E[EE^j_i(t)]\|_2.$$  \hspace{1cm} (2)

To evaluate the effect of $i$th input to $j$th output over all time instead of at a particular event, the $L_2$ norm is preferred over other norms.

In what follows, the computational cost is defined as the number of evaluations of a function, or the number of independent time-domain simulations in the case of ODEs. To compare the convergence characteristics of all these sensitivity techniques, we need to define an absolute difference between estimated and reference solutions of $E[EE^j_i]$ and $\sigma[EE^j_i]$, normalized by the absolute value of the reference solution. For ODEs, as noted above the absolute value is replaced by the $L_2$ norm to average $E[EE^j_i]$ and $\sigma[EE^j_i]$ over a specified time interval. For consistency, the reference solution for any given case must have a much higher accuracy than the points that make up the convergence plot, and in fact the reference trajectory’s accuracy has to be inferred from the existing trends when it is not known exactly. In this work, we keep the reference results about one order of magnitude better than the points we use to show convergence.

The normalized errors, based on the absolute differences, are:

$$\epsilon_{i,\text{mean}} = \frac{|E[EE^j_i] - E[EE^j_{i,\text{ref}}]|}{E[EE^j_{i,\text{ref}}]} \hspace{1cm} (3)$$

$$\epsilon_{i,\text{var}} = \frac{|\sigma[EE^j_i] - \sigma[EE^j_{i,\text{ref}}]|}{\sigma[EE^j_{i,\text{ref}}]} \hspace{1cm} (4)$$

Sometimes it is of interest to assess the overall sensitivity of an output to all the input channels; the root-mean-square over the inputs is given as:

$$\text{RMS}(\epsilon^j_{i,\text{mean}}) = \left[ \frac{1}{d} \sum_{i=1}^{d} (\epsilon^j_{i,\text{mean}})^2 \right]^{\frac{1}{2}}$$  \hspace{1cm} (5)$$

$$\text{RMS}(\epsilon^j_{i,\text{var}}) = \left[ \frac{1}{d} \sum_{i=1}^{d} (\epsilon^j_{i,\text{var}})^2 \right]^{\frac{1}{2}}$$  \hspace{1cm} (6)

In the Variance method, we replace $(E[EE^j_i], \sigma[EE^j_i])$ with $(V[EE^j_i], IEE^j_i)$ in the convergence computations.

III. COMPARISON OF SENSITIVITY METHODS FOR SAMPLE PROBLEMS

To demonstrate how the sensitivity algorithms are applicable to both functions and systems of ODEs, and to evaluate their relative convergence performance, in this section we systematically compute the parametric sensitivity and interaction for a function with twelve input parameters (modified from Morris’s original function [14]), for a nonlinear first-order ODE with an exact solution, and for a simplified induction machine model. All numerical computations are performed with Microsoft C++ compiler on an Intel Pentium 4 3.0GHz Processor.
A. Modified Morris Function

A function with twelve stochastic inputs, modified from the original test problem with twenty inputs [14], is used here for comparing the performance of all sensitivity algorithms. This modified Morris function is useful because the inputs have strong coupling among them:

\[
y = \beta_0 + \sum_{i=1}^{n} \beta_i x_i + \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \beta_{i,j} x_i x_j + \sum_{k=1}^{n-2} \sum_{j=1}^{j-1} \sum_{i=1}^{i=1} \beta_{i,j,k} x_i x_j x_k,
\]

\[(7)\]
\[
\beta_0 = 1,
\beta_i = \begin{cases} 20, & \text{for } i = 1, \ldots, 10 \\
-15, & \text{for } i, j = 1, \ldots, 6,
\beta_{i,j} = -10, & \text{for } i, j, k = 1, \ldots, 5,
\end{cases}
\]

(8)

The \(x_i\) are in the uniform distribution over the unit cube, the rest of the \(\beta_i\) and \(\beta_{i,j}\) are drawn from normal distributions with unity variance and zero mean. The coefficients of the remaining \(\beta_{i,j,k}\) are set to zero.

The Morris method with \(\Delta = 8/15\), and the MC Sampling and Collocation methods with \(\Delta = 1/2\) classify the inputs’ sensitivities via \(E[EE_i]\) and \(\sigma[EE_i]\) into four distinct groups: (1,2,3,4,5), (6), (7,8,9,10), and (11,12) as shown in Figure 1. The first group, (1,2,3,4,5), has high sensitivity and strong coupling, whereas the third group of inputs, (7,8,9,10), exhibits a high elementary (linear) effect with a minor interaction with the other inputs. The (11,12) group has a negligible effect on the output, and (6) has an intermediate score on these measures. Also in this figure, the \(VEE_i\) and \(IEE_i\) metrics of the Variance method rank these twelve inputs in the same order as do \(E[EE_i]\) and \(\sigma[EE_i]\) for the gradient-based method. Thus, all four algorithms identify the inputs’ sensitivity similarly for this nonlinear function.

Results from the Morris method with \(r = 10^5\), QMC Sampling with \(N = 5 \times 10^6\), and the Variance+QMC method with \(N_c = 20\) and \(N = 5 \times 10^6\) are used as reference solutions. With regards to the convergence performance in Figure 2, the Collocation and Variance methods appear to offer better results than the Morris, MC, and QMC Sampling methods, in the low accuracy region. However, these curves are relatively flat, and the other methods attain better performance, at the higher accuracies shown. The QMC and Variance+QMC methods are clearly the superior approaches here.

B. Linear first-order ODE

Next, we consider a nonlinear first-order ODE. For this system, we can derive analytically the elementary effect and the standard deviation ratio of the output over input, which are used as reference solutions for our sensitivity indices. The system is:

\[
\frac{dy}{dt} = -ky \text{ with } y(0) = y_0 = 2,
\]

(9)

FIGURE 1. For the modified Morris function with 12 inputs and \(x_i \in U[0,1]\): the mean and standard deviation of \(EE_i\) from the Morris method with \(p = 16\) and \(r = 8,000\) to 10,000 (first row), from the MC Sampling with \(\Delta = 1/2\) and \(N = 8,000\) to 10,000 (second row), the Collocation method (FPCM) with \(\Delta = 1/2\) and \(N_c = 12\) to 14 (third row), and the Variance method with \(N_c = 12\) to 14 (fourth row).
The exponential series up to 5 terms, i.e.,

$$\frac{|\frac{dy}{dt}|}{\sigma_k} = \frac{2(\sigma_k t)^2}{\sqrt{2} + e^{2\sigma_k t}(\sigma_k t - 1) - e^{-2\sigma_k t}(1 + \sigma_k t)}$$

(11)

However, the absolute ratio of the local derivative over the statistical solution varies as time progresses. The bigger the magnitude of $\sigma_k$ is, the larger the deviation of $|\frac{dy}{dt}|/\sigma_k$ as a function of time becomes.

![Figure 2](image-url.png)

**FIGURE 2.** For the modified Morris function with twelve inputs and $x_i \in U[0, 1]$: RMS($\epsilon_{\text{mean}}$) (Upper) and RMS($\epsilon_{\text{var}}$) (Lower) convergence versus computational time, using the Morris method, MC and QMC Sampling, Collocation methods, Variance, and Variance+QMC methods.

where the decay rate coefficient $k$ is assumed to be a uniform random variable with mean value $k = 5$ and standard deviation $\sigma_k = 0.4$. The deterministic solution of this equation (i.e., with $k$ fixed) is $y(t) = y_0 e^{-kt}$. The local derivative of $y(t)$ with respect to $k$ is then $dy/dk = -y_0 e^{-kt}$. The ratio of $\sigma_y$ to $\sigma_k$ is given by

$$\frac{\sigma_y}{\sigma_k} = \sqrt{\frac{y_0^2 e^{-2kt}}{2} \left( \frac{e^{-2\sigma_k t} - e^{2\sigma_k t}}{2\sigma_k t} + \frac{(e^{\sigma_k t} - e^{-\sigma_k t})^2}{2\sigma_k^2 t^2} \right)}.$$  

(10)

The mean sensitivity index $E[EE_k]$ from the gradient-based methods is plotted overlaying the local derivative, $dy/dk$, in Figure 3. Similarly, the stochastic sensitivity index $VEE_k$ from the Variance method is directly superimposed on the closed-form solution for $\frac{\sigma_y}{\sigma_k}$, in Figure 3. Both the gradient-based and Variance methods agree well with the analytical solutions. Notice that a scaling factor between the absolute local derivative and the statistical solution at the initial point ($t = 0$) is $\sqrt{3}$. This scaling factor can be obtained analytically from the ratio of $|\frac{dy}{dt}|$ over $\sigma_y$ by using an approximation of

![Figure 3](image-url.png)

**FIGURE 3.** Linear ODE $dy/dt = -ky$ with $\sigma_k = 0.4$: the mean of $EE_k$ from the Morris method with $p = 16$, MC Sampling with $\Delta = 1$, and Collocation with $\Delta = 1$ methods are compared with the absolute local derivative of $y$ with respect to $k$ (Upper) and the $VEE_k$ from the Variance method are compared with the statistical solution $\frac{\sigma_y}{\sigma_k}$ (Lower).
C. An open-loop induction machine with an infinite bus

In this section, the sensitivity algorithms are tested on a 200-hp Induction Machine (IM) connected to an infinite bus, as shown in Figure 5. Although simplified here, the induction machine is relevant as a building block in the more complicated model presented in the next section. The detailed derivation of system modeling and parameters can be found in Appendix II [13], and stochastic simulation results are described in [15]. In this problem, all the parameters are strongly coupled with other parameters, as well as with state variables. We assume all parameters to be independent random variables with ±10% variation from their nominal values.

The mean 2-norm sensitivities ($ES_{2,(j,i)}$) are shown as intensities in a table of input and output parameters in Figure 6. Also, Figure 6 shows the $\sigma[E_{k}]$ versus $E[E_{k}]$ plot of $\psi'_{dr}$. When $E[E_{k}]$ is large, the $\sigma[E_{k}]$ also tends to be large, and the loops indicate a very dynamic interaction of these parametric measures. The $\psi'_{dr}$ output is the most sensitive, particularly with respect to machine’s electrical parameters and the rotor inertia. In fact, it is well known that an open-loop induction machine’s behavior is highly sensitive to rotor resistance and inertia fluctuations.

Figure 7 shows the convergence performance for the sensitivity techniques in the ten-dimensional parametric space. Again using the $RMS(\epsilon_{\text{mean}})$ and $RMS(\epsilon_{\text{var}})$ as the convergence indices, we see that the Morris and MC Sampling methods converge faster than the Collocation method because collocation methods are computationally intensive in high dimensions. The Multi-Element technique [15] can improve the convergence of the Collocation technique, however, and
the reader is referred to [15].
Further comparison among the gradient-base methods confirms that high accuracy in the sensitivity indices can be best obtained with the QMC Sampling. The poorer convergence rates of the Collocation and Variance methods are similar to each other. On the other hand, a combination of the full-grid Collocation and Monte Carlo techniques in the Variance+QMC method can lead to an effective convergence rate. The reference solutions for the Morris and other gradient and Variance methods are from the Morris method with \( r = 10^5 \), the QMC sampling with \( N = 5 \times 10^5 \), and the Variance+QMC with \( N_c = 20 \) and \( N = 5 \times 10^4 \), respectively.

### FIGURE 7.
The convergence characteristics of \( RMS(\epsilon_{\text{mean}}) \) (Upper) and \( RMS(\epsilon_{\text{var}}) \) (Lower) using the Morris, MC and QMC Sampling, Collocation, Variance, and Variance+QMC methods is plotted versus the computational time.

### IV. AC POWER DISTRIBUTION WITH OPEN-LOOP PROPULSION SYSTEM

We next perform sensitivity analysis for a large-scale multirate dynamical system consisting of AC power generation and propulsion subsystems in the all-electric ship [11]. The response of this system occurs on two different time scales, associated with the electrical and mechanical time constants of the synchronous generator and the induction motor. A one-line diagram is shown in Figure 8. This system consists of a 59 kW Synchronous Machine (SM), an IEEE type 2 exciter/voltage regulator [10], a 50-hp Induction Machine (IM), and a three-phase RC bus with harmonic filter; all the parameters can be found in [11], see also Appendix II. There are a total of 24 uncertain parameters listed in Table 3 (twelve in the synchronous machine and exciter/voltage regulator, seven in the induction machine, and five in the RC bus connecting between the synchronous and induction machines), and 25 state variables as listed in Table 2. The mechanical torque load is modeled as being proportional to the motor speed squared, similar to a propeller load. For our calculations, we assume all 24 uncertain parameters to be independent uniform random variables with ±30% variation from their nominal values. This particular model was also studied in detail using stochastic simulation techniques based on the full- and sparse-grid Probabilistic Collocation methods in [15].

### FIGURE 8.
One-line diagram of the AC power distribution system and open-loop induction machine, with 25 state variables and 24 uncertain parameters.

In this simulation, the synchronous machine is initially operating in its steady state at rated speed and voltage; the induction machine is turned on at time zero seconds. The electrical transient responses die out within the first second, while the responses of the mechanical subcomponents approach the steady state within about six seconds.

Based on the prior convergence studies, we employed here only the QMC Sampling method with \( \Delta = 1 \), and carried out separate analyses for \( t \in [0, 1] \) and \( t \in [0, 12.4] \) seconds, where the electrical and mechanical time constants dominate, respectively. The normalized sensitivity time traces of the a-phase harmonic filter’s current, \( I_a \), and the induction machine’s rotor speed, \( \omega_r \), up to one second are shown in Figure 9 for the five most sensitive parameters. Both outputs are sensitive to both electrical and mechanical parameters; however, the \( I_a \) sensitivity indices approach (small) steady state values more quickly.

The average sensitivity indices in the short time scale are summarized in the \( ES_2(j,k) \) plot of Figure 10. The three-phase currents \( (I_a, I_b, I_c) \) of the harmonic filter in the RC bus are the most sensitive output variables, and they are highly sensitive to \( C_f \) and \( L_f \), because the harmonic filter is tuned to reduce high harmonic frequency and the RC bus is subject to the high-frequency start-up current of the induction machine. Moreover, most state variables are sensitive to the induction machine’s parameters, which implies a direct interdependency among the synchronous and induction machines, and the RC bus.

For \( t \in [0, 12.4] \), where the mechanical time constant is dominant, the normalized sensitivity trajectories of \( I_a \) and \( \omega_r \) are shown in Figure 11. The five most influential parameters...
We can see that in particular the sensitivity of $I_a$ to $r'_{r2}$ and $J$ after 2 seconds confirms strong interactions between the RC bus and the induction machine. The sensitivity time traces for $\omega_r$ to $r'_{r2}$ and $J$ are much larger than seven times of those seen in the short time frame; they then approach a zero steady-state value within three mechanical time constants. We also note that the induction machine’s rotor speed also becomes more sensitive to the load coefficient, $\alpha_{load}$, after two seconds, and this implies that some parameters have a strong influence at different time scales.

From the $ES_{2,(j,i)}$ intensities in Figure 12, we observe that only three parameters - $r_{fd}$ of the synchronous machine, and $r'_{r2}$ and $J$ of the induction machine - have significant impact in this longer time frame. These three are particularly influential because: (1) the rotor inertia directly governs the mechanical time constant, (2) the rotor resistance of the
induction machine has a direct influence on the generated rotor flux and motor operation, and (3) the rotor field winding of the synchronous machine, which is connected to the voltage feedback from the exciter/voltage regulator, can amplify the propagation of uncertainties. Therefore, a small variation in \( r_{fd} \) of the synchronous machine can lead to a large fluctuation in the bus voltage.

![Image](image_url)

FIGURE. 12. For \( t \in [0, 12.4] \) second, the \( \mathcal{E}S_{2,(j,i)} \) plot using the QMC Sampling method with \( N = 3,000 \).

V. SUMMARY

All of the sensitivity analysis techniques proposed here can be effective to identify the most influential parameters in a large-scale, complex power system, and particularly in a multi-rate dynamical system such as the all-electric ship, which has to maintain operational capabilities under parameter and loading uncertainties, as well as reconfiguration. The examples here, while simplified, show very strong interdependency among machines, and accentuate the fact that responses occur on several time scales.

Both the gradient-based methods - MC QMC Sampling and Collocation - and the variance-based methods - Variance and Variance+MC methods - identically rank the influential parameters and detect the most coupled parameters, in a manner that is consistent with the Morris method, but may be performed at a reduced computational cost. Combining the advantage of Gauss-quadrature integration and the Quasi-Monte Carlo technique, the QMC and Variance+QMC methods yield the best convergence performance in the cases studied here.

APPENDIX I

SENSITIVITY ANALYSIS METHODS

A. Morris Method

The Morris method considers the One-At-a-Time \( EE_i^j \) to identify the significant elementary and interaction effects of input parameters with only a few evaluations of \( EE_i^j \). The basic methodology of this approach is to randomly select an initial condition on the grid points and construct a randomized trajectory along this grid structure in a high-dimensional input space for \( r \) trials. Thus, the mean and standard deviation of \( EE_i^j \) with respect to the number of trial \( (r) \) represent the elementary effect of \( i \)th input and interaction of other inputs with the \( i \)th input. Originally, all input parameters in the Morris method [14] are assumed to be independent uniformly distributed; nevertheless, the normal distribution can be applicable to parameters in this method as well [5]. The Morris method becomes more efficient when \( d \gg r \) [14].

First, each input dimension of the \( d \)-dimensional hypercube is divided into a \( p \)-level grid, \([0, 1/(p − 1), 2/(p − 1), \ldots, 1]\), such that the initial condition, \( \mathbf{x}^* \), can be randomly assigned to one of these grid points. According to Morris, the value of \( \Delta \) is set as \( p/(2(p − 1)) \) using an even \( p \) value and \( p > 2 \) such that \( \Delta \) optimally covers the \( d \)-dimensional \( p \)-level space with an equal probability. It is clear that the randomized trajectory is contained within the range of input variation. The detailed procedure to construct the feasible One-At-a-Time randomized trajectory can be found in [14], [19]. We show the deterministic trajectory (\( \Delta \mathbf{B} \)) and one of the \( r \) randomized trajectories (\( \Delta \mathbf{B}^r \)) for \( p = 4 \) in Figure 13.

![Image](image_url)

FIGURE. 13. Morris Method: Trajectories from \( \Delta \mathbf{B} \) and \( \Delta \mathbf{B}^r \) for \( p = 4 \) and \( \Delta = 2/3 \).

The total computational cost consists of two parts: generating the randomized trajectory with cost on the order of \( O(r \times d) \) and evaluating the \( EE_i^j \) for one output using the randomized trajectories with cost on the order of \( O(r \times (d + 1)) \). Note that, as we increase the \( p \)-level in this \( d \) dimensional space, the value of \( \Delta \) approaches \( 1/2 \) and the probability of the randomize trajectory staying with the interior domain approaches 1. Nevertheless, the computational cost for constructing the randomized trajectories increases tremendously as \( p \) increases to cover the parametric space with an equal probability.

B. Monte Carlo Sampling Method

Instead of computing the statistics of the \( EE_i^j \) from the randomized trajectories on the \( p \)-level grid as in the Morris method, a Monte Carlo (MC) or Quasi-Monte Carlo (QMC) Sampling methods can be used to randomly generate the \( N \) initial conditions in the \( d \)-dimensional inputs, and then the
elementary effect in each direction can be computed at these \( N \) random initial points. Similar to the Morris method, the mean of each \( i \) elementary effect or \( E[EE^i_i] \) with respect to \( N \) realizations can be directly used to rank input parameters. In addition, \( \sigma[EE^i_i] \) with respect to \( N \) can specify the respective influences of inputs’ interaction and nonlinearity on the output.

To demonstrate this methodology, Figure 14 shows 9 realizations of random initial conditions and directions in the \( EE^i_i \) computation with fixed \( \Delta \) in a three-dimensional input space. The total evaluation of output for \( d \) directions is on the order of \( O(N \times (d + 1)) \) for the \( d \) parameters and inputs. Therefore, the accuracy of \( E[EE^i_i] \) and \( \sigma[EE^i_i] \) depends on the convergence characteristic of the Monte Carlo method, approximately \( 1/\sqrt{N} \) and \( 1/N \) for pseudo- and quasi-random sampling techniques, respectively. This rate of convergence is less sensitive to the parameter dimension, which is attractive for metamodels. The main advantages of this approach over the Morris method are that the approximated gradient computation is not restricted only on the \( p \)-level grid structure. Thus, the method inherits the advantages for high-dimensionality of Monte Carlo methods.

![Figure 14](image)

**FIGURE. 14.** Monte Carlo Sampling Method: With \( N = 9 \), the random direction and initial condition in each direction of a three-dimensional input space are used for computing the \( EE^i_i \) for \( i = 1,2,3 \) with a fixed \( \Delta \).

Applying this technique to analyze the sensitivity of ODEs requires solving only \( N \) problems. At each time step, we perturb the system inputs one at a time with a fixed \( \Delta \). Similar to a maximum limit of the \( \Delta \) magnitude in the Morris method, which equals 1/2 for an input range between \([0,1]\), we assign the \( \Delta \) magnitude to be half of the parameter-variation range. If \( \Delta \) is greater than 1/2 of the input range, the distribution of \( EE^i_i \) might be misleading due to possible strong nonlinearity present in the system, and the value of \( x_i + \Delta \) in \( EE^i_i \) computation can exceed the input domain. On the other hand, if the magnitude of \( \Delta \) becomes too small, the finite-difference value of \( EE^i_i \) might blow up. An adaptive \( \Delta \)-magnitude approach can be further investigated. However, if the value of \( x_i \pm \Delta \) is outside the input domain, the sign of \( x_i \pm \Delta \) must be reversed.

### C. Collocation Method

To further improve the accuracy and efficiency of this parameter screening technique, the initial condition of the \( EE^i_i \) computation can be selected at specific locations – the collocation points – instead of at the random sampling points. Thus, the numerical integration technique, based on the quadrature rule \([21],[22]\), should provide an advantage in computing the statistics of \( EE^i_i \). The procedure for this approach is similar to the Monte Carlo sampling method. First, we specify a distance of \( \Delta \) in computing the \( EE^i_i \) and the number of collocation points defined by the abscissas of the Legendre polynomials for the uniform variation between \([0,1]\) or \([-1,1]\). Second, the elementary effect in each \( i \)-th input direction is calculated at the collocation points with a random direction of \( \Delta \). Lastly, the mean and standard deviation of \( EE^i_i \) with respect to the number of collocation points \((N_c)\) are obtained from the Gauss-Legendre quadrature \([16]\) – we refer to this as the “full-grid collocation method” (FPCM). Again, \( E[EE^i_i] \) and \( \sigma[EE^i_i] \), respectively, represent the elementary effect and the nonlinear and coupling effect of inputs. According to the Gauss-quadrature rule, we can expect an exponential convergence rate in statistical results when the system is evaluated with parameters at the collocation points. As shown in the next section, the statistical convergence can be algebraic in marginally resolved cases. To illustrate the concept of this method, Figure 15 shows how we combine the approximated gradient calculation with the collocation method. The computational cost of using this technique depends on the input dimension, which is \( O(N_c^d \times (d + 1)) \), where \( N_c \) is a number of collocation points per random dimension.

![Figure 15](image)

**FIGURE. 15.** Collocation Method: With 8 points or \( N_c = 2 \), the approximated gradient, \( EE^i_i \) with a fixed \( \Delta \), is computed at the collocation points with a random direction in each dimension of the three-dimensional parametric space.

### D. Variance Method

The Variance method introduced here directly takes advantage of the efficiency and accuracy of the collocation method to identify each input sensitivity and input interaction. This method relies on a variation of the output when only one input is a random variable instead of using the approximated gradient to measure the One-At-a-Time sensitivity. Note that
we use the standard deviation as a sensitivity index when we refer to the Variance method. First, let us define the elementary effect (V\(EE_i^j\)) of \(i\) input on \(j\) output, which is closely related to the mean of \(EE_i^j\).

\[
V\(EE_i^j\) = E_{x_i} (\sigma_{x_i} [y^j (x)])
\]

where \(\sigma_{x_i}\) denotes the standard deviation of the \(j\)th output, \(y^j (x)\), when only \(x_i\) is a random variable and the other inputs are fixed at the specified points in the \(d - 1\) input dimension.

\[
I\(EE_i^j\) = \sigma_{x_i} (\sigma_{x_i} [y^j (x)])
\]

where \(\sigma_{x_i}\) denotes the standard deviation of all other inputs except the \(x_i\) input. The magnitude of \(I\(EE_i^j\)\) can only specify the coupling of the \(i\)th parameters without taking into account the nonlinearity of the \(x_i\) term. The Gauss quadrature is used for both the \(\sigma_{x_i}\) and \(E_{x_i} (\sigma_{x_i} [y^j (x)])\) integral calculation. Therefore, an exponential convergence of both \(V\(EE_i^j\)\) and \(I\(EE_i^j\)\) can be expected with the use of the collocation method in each dimension for small input dimensions. The evenly distributed collocation points of the full-grid collocation method allow for a thorough exploration of the interaction of the \(x_i\) input with the other inputs. When \(d\) is large, instead of computing \(E_{x_i} (\sigma_{x_i} [y^j (x)])\) and \(\sigma_{x_i} (\sigma_{x_i} [y^j (x)])\) in \((d - 1)\) dimensions using the quadrature rule in Eqn. (13) and (15), these two quantities can be computed with the Monte Carlo technique, referred to this as the Variance+MC method, to accelerate the solution convergence.

Figure 16 and 17 demonstrates how to obtain the elementary and coupling effects from the standard deviation of each \(x_i\) input in the case of three parameters. The computational costs for Variance and Variance+MC methods are approximately \(O(N_e^d)\) and \(O(N_e \times N \times d)\), respectively. Nevertheless, this Variance method yields a more accurate sensitivity solution than the two previous methods for the small input dimensions due to an error only from the Gauss quadrature. In low dimension, the variance method is fast because evaluations can be reused to cover all the combinations needed and the quadrature rule is well suited for a low dimensional integral.

APPENDIX II
SYSTEM MODELING AND PARAMETERS OF THE OPEN-LOOP INDUCTION MACHINE WITH AN INFINITE BUS

We briefly describe the formulation of a general mathematical model for the symmetrical three-phase squirrel-cage induction machines with four poles and three-phase windings in both the stator and rotor connected in a wye configuration.

The governing equations are written in the \(qdl\) synchronous reference frame (denoted by superscript \(e\)). The voltage equations of the stator and rotor windings can be written as

\[
v_q^{e,q} = -\mathbf{r}_s \mathbf{i}_q^{e,q} + \frac{\omega_e}{\omega_b} \mathbf{T}_1 \psi_q^{e,q} + \frac{p}{\omega_b} \psi_{qd0}^{e,q}\]

\[
v_{qd0}^{e,r} = \mathbf{r}_r \mathbf{i}_q^{e,q} - \frac{\omega_e}{\omega_b} T_1 \psi_q^{e,q} + \frac{p}{\omega_b} \psi_{qd0}^{e,q}\]

where \(v_q^{e,q}, i_q^{e,q}, \psi_q^{e,q}\) and \(v_{qd0}^{e,r}, i_{qd0}^{e,r}, \psi_{qd0}^{e,r}\) are the stator and rotor variables of voltage, current, and flux, expressed in a vector form as \(f_{qd0}^{e,q} = [f_{qs}^{e,q}, f_{ds}^{e,q}, f_{ds}^{e,q}]^T\) and \(f_{qd0}^{e,r} = [f_{qr}^{e,q}, f_{dr}^{e,q}, f_{dr}^{e,q}]^T\), respectively. The resistance matrices \(\mathbf{r}_s\) and \(\mathbf{r}_r\) are diagonal matrices \([r_s, r_s, r_s]\) and \([r_r, r_r, r_r]\). Here, \(\omega_e\) and \(\omega_b\) are the rotor angular and base velocities; the synchronous speed \(\omega_c\) is the same as \(\omega_b\) in the absence of a controller. The positive direction of stator current is assumed to be outward.
from the stator winding. The prime symbol denotes that the rotor variables are scaled by a stator to rotor turn ratio.

The equations of flux linkage per second are

\[
\psi_{q'd0s} = -X_{lq} \psi_{q'd0s} + \psi_{q'mq} \\
\psi_{q'd0r} = X_{lq} \psi_{q'd0r} + \psi_{q'mq},
\]

where the flux leakage matrices \(X_{ls}\) and \(X_{lr}\) are diag\([x_{ls}, x_{ls}, x_{ls}]\) and diag\([x_{lr}, x_{lr}, x_{lr}]\). We write \(\psi_{q'mq} = -X_{lq} \psi_{q'd0s} + A_q \psi_{q'd0r}\), where \(X_{ls} = \text{diag}[x_{ls}, x_{ls}, x_{ls}]\) with \(x_{ls} = (x_m x_{lr}) / (x_m + x_{lr})\), and \(A_q = \text{diag}[x_{lr}, x_{lr}, x_{lr}]\), which is the mutual inductance of the stator and rotor.

The dynamics of the mechanical subsystem can be written as

\[
p = \frac{\omega_r}{2H}(T_e - T_L),
\]

where \(T_e = \psi_{q'd0s}' - \psi_{q'd0r}'\) and \(T_L\) are the electromagnetic and load torques, respectively. The rotor inertia (in seconds) is \(H\).

**TABLE 1** Parameters of the 200 Hp induction machines in per unit.

<table>
<thead>
<tr>
<th>(r_s)</th>
<th>(x_{ls})</th>
<th>(x_m)</th>
<th>(x_{lr})'</th>
<th>(r_e')</th>
<th>(H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0655</td>
<td>3.225</td>
<td>0.0655</td>
<td>0.0261</td>
<td>0.922</td>
</tr>
</tbody>
</table>

**TABLE 2** State variables of the synchronous generator/exciter (1-10), the induction machines (20-25), and the RC bus (11-19) with \(V_{base} = 450\) V.

| | q-axis stator flux linkage | q-axis damper winding flux linkage | d-axis stator flux linkage | d-axis field winding flux linkage | d-axis damper winding flux linkage | 0-axis stator flux linkage | Power angle | Exciter Voltage | Exciter State | a-phase bus voltage | b-phase bus voltage | c-phase bus voltage | a-phase harmonic filter current | b-phase harmonic filter current | c-phase harmonic filter current | a-phase harmonic filter voltage | b-phase harmonic filter voltage | c-phase harmonic filter voltage | q-axis stator flux linkage | d-axis stator flux linkage | d-axis rotor flux linkage | 0-axis stator flux linkage | Rotor Rotational Speed |
|---|-----------------|-----------------|--------------------|-----------------|----------------|-----------------|-------------|-----------------|-----------------|-------------------|-------------------|-------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-------------------|-----------------------------|
| 1 | \(\psi_{q's}\) | \(\psi_{h1l}\) | \(\psi_{e2}\) | \(\psi_{ef}\) | \(\psi_{fd}\) | \(\psi_{fd}\) | \(\theta\) | \(V_F\) | \(V_{em}\) | \(V_{bn}\) | \(V_{cm}\) | \(I_a\) | \(I_b\) | \(I_c\) | \(V_{fa}\) | \(V_{fb}\) | \(V_{fc}\) | \(\psi_{qsr}\) | \(\psi_{dsr}\) | \(\psi_{dr}\) | \(\omega_r\) |

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